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LETTER TO THE EDITOR

Anyon winding numbers from the Chern–Simons interaction

T Haugset and F Ravndal

Institute of Physics, University of Oslo, N-0316 Oslo 3, Norway

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Abstract. Integrating out the gauge field in the Chern–Simons interaction of anyons, we recover in the non-relativistic limit the winding number in the action for particles with fractional statistics.

Leinaas and Myrheim [1] in their seminal paper pointed out that quantum particles in $2+1$ dimensions can obey fractional statistics which is intermediate between ordinary Bose–Einstein and Fermi–Dirac statistics. The wavefunctions will in general be multi-valued, depending on the winding of each particle around the others in the plane of motion.

For a non-relativistic system of N such particles the dynamics is given by the Lagrangian

$$L = L_0 + \frac{\theta}{\pi} \sum_{n<m}^{N} \dot{\phi}_{nm} \tag{1}$$

where the first term describes the classical system. In the last term, the time derivative $\dot{\phi}_{nm}$ of the multi-valued, polar angle

$$\phi_{nm} = \tan^{-1} \frac{y_n - y_m}{x_n - x_m} \tag{2}$$

describes the winding of particle n with coordinates $\boldsymbol{r}_n = (x_n, y_n)$ around particle m with coordinates $\boldsymbol{r}_m = (x_m, x_y)$. This topological term in the Lagrangian is proportional to the statistics parameter θ which is $0 \bmod 2\pi$ for bosons and $\pi \bmod 2\pi$ for fermions.

A physical model of particles with fractional statistics has been proposed by Wilczek [3] and called anyons. They are endowed with statistical charge and magnetic flux so that the multi-valuedness of the wavefunction is due to the Aharonov–Bohm effect [2]. Introducing a vector potential $A_\mu(x)$ for the statistical field, a system of anyons can now be described by the Lagrangian [4]

$$L = L_0 + \int \mathrm{d}^2x \left(J^\mu(x) A_\mu(x) + \frac{\kappa}{2} \varepsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda \right) \tag{3}$$

where the first term in the integral gives the coupling to the conserved particle current

$$J_\mu(x) = \sum_{n=1}^{N} \int \mathrm{d}\tau\, \dot{x}_n^\mu(\tau) \delta^3(x - x_n(\tau)) \tag{4}$$

and the last term is the Chern–Simons interaction [6] with strength κ.

In most studies of particles with fractional statistics the two Lagrangians (1) and (4) are usually considered to be equivalent. Since we are not aware of any explicit demonstration of this equivalence in the literature, we will show here that it is quite straightforward except for a subtle point of particles formally winding around themselves.

Starting with the Lagrangian (4), the partition function is given by the functional integral

$$Z=\int \mathrm{D}x \int \mathrm{D}A\, \mathrm{e}^{-\int \mathrm{d}\tau L} \tag{5}$$

where we now work in Euclidean spacetime with coordinates $x^\mu=(c\tau, x, y)$. The integral is quadratic in the vector potential A_μ and can therefore easily be integrated out [7]. It gives the factor

$$\tilde{Z}=\exp\left(\tfrac{1}{2}\int \mathrm{d}^3x\, J^\mu(k) M^{-1}_{\mu\nu}(k) J^\nu(-k)\right) \tag{6}$$

where $J^\mu(k)$ are the Fourier components of the particle current (4) and

$$M^{-1}_{\mu\nu}(k)=\frac{-\mathrm{i}}{\kappa}\,\varepsilon_{\mu\nu\sigma}\frac{k^\sigma}{k^2} \tag{7}$$

when leaving out terms proportional to k_μ since $k_\mu J^\mu(k)=0$. Using the integral

$$\int \frac{\mathrm{d}^3k}{(2\pi)^3}\frac{\mathrm{e}^{\mathrm{i}kx}}{k^2}=\frac{1}{4\pi|x|} \tag{8}$$

we can write the result of the functional integration as

$$\tilde{Z}=\exp\left(-\frac{1}{8\pi\kappa}\sum_{n,m}^{N}\int \mathrm{d}x_n^\mu \int \mathrm{d}x_m^\nu\, \frac{\varepsilon_{\mu\nu\lambda}(x_n^\lambda - x_m^\lambda)}{|x_n-x_m|^3}\right) \tag{9}$$

where the terms in the exponent are the Gauss linking numbers of the different particle paths.

We now first consider terms with $n\neq m$. They have the form

$$I_{nm}=c\int \mathrm{d}\tau_n \int \mathrm{d}\tau_m \frac{(\dot{x}_n\dot{y}_m-\dot{y}_n\dot{x}_m)(\tau_n-\tau_m)+\varepsilon_{ab}(x_n^a-x_m^a)(\dot{x}_n^b-\dot{x}_m^b)}{[c^2(\tau_n-\tau_m)^2+r_{nm}^2]^{3/2}} \tag{10}$$

with $r_{nm}^2=(x_n-x_m)^2+(y_n-y_m)^2$. The Latin indices a and b run from 1 to 2 and ε_{ab} is the antisymmetric symbol in two dimensions. In the non-relativistic limit $c\to\infty$ we see from the form of the denominator that the leading contributions to the line integrals are coming from points where $\tau_n\approx\tau_m$. Under the integration over τ_m we can therefore consider the coordinates and velocities to be constant. The remaining integral

$$I_{nm}^{\mathrm{NR}}=c\int \mathrm{d}\tau_n \int \mathrm{d}\tau_m\, \varepsilon_{ab}\frac{(x_n^a-x_m^a)(\dot{x}_n^b-\dot{x}_m^b)}{[c^2(\tau_n-\tau_m)^2+r_{nm}^2]^{3/2}} \tag{11}$$

is then given by

$$\int \frac{\mathrm{d}x}{(ax^2+b)^{3/2}}=\frac{x}{b\sqrt{ax^2+b}}. \tag{12}$$

At zero temperature, the lower and upper limits of the integral are at $\pm\infty$ and we find

$$I_{nm}^{\mathrm{NR}}=2\int \mathrm{d}\tau_n\, \varepsilon_{ab}(x_n^a-x_m^a)(\dot{x}_n^b-\dot{x}_m^b)/r_{nm}^2. \tag{13}$$

Since the integrand is just the time derivative of the polar angle (2) we see that this part of the functional integral with $1/2\kappa = \theta$ gives

$$\tilde{Z} = \exp\left(-\frac{\theta}{\pi} \int \mathrm{d}\tau \sum_{n<m} \dot{\phi}_{nm} \right) \tag{14}$$

which proves the equivalence with the Lagrangian (1).

The diagonal terms with $n = m$ in (9) are seen to be formally divergent and must be regularized. For non-relativistic particles as considered here, Jackiw [5] has shown that their contributions can in fact be set equal to zero. On the other hand, Polyakov [8] and others [9] have shown that these diagonal terms are finite for relativistic particles and related to geometrical properties of the particle paths having to do with their spin degrees of freedom.

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